Dynamics of inhomogeneous one–dimensional coupled map lattices

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Abstract

We study the dynamics of one–dimensional discrete models of one–component active medium built up of spatially inhomogeneous chains of diffusively coupled piecewise linear maps. The nonhomogeneities ("defects") are treated in terms of parameter difference in the corresponding maps. Two types of space defects are considered: periodic and localized. We elaborate analytic approach to obtain the regions with the qualitatively different behaviour in the parametric space. For the model with space–periodic nonhomogeneities we found an exact boundary separating the regions of regular and chaotic dynamics. For the chain with a unique (localized) defect the numerical estimate is given. The effect of the nonhomegeneity on the global dynamics of the system is analyzed.

Key words: coupled map lattices, nonhomogeneities, chaos, Chebyshev polynomials
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1 Introduction

One of the effective methods of analysis of several systems is based on their representation by discrete models. For this one can use the space discretization as well as the time one. Using the space discretization the initial (continuous) system is modelled by a finite set of elements that are coupled according to some rules, which follows from the nature of the system of interest. Each element of these then occurs to be some (small) dynamical system which

may be described by a (small) set of dynamical variables. If the time is also discretized, i.e. if the evolution of these dynamical subsystems is considered at discrete time–steps, then such subsystems are treated as maps. In this case spatio-tempotal models are said to be *lattice* of coupled maps.

The examples of application of the space–time discretization in studies of the nonlinear media are numerous. Here we mention the problems related to chaos and noise in distributed systems [1,2,3,4], to description of the excitable media [4,5] and some other similar problems (see e.g. [6,7,8,9,10,11,12,13,14] and references therein). Moreover, any numerical analysis of a continuous system has in fact to be reduced to the study of space–time discretized model.

In principle, a variety of different space–time lattices may be used. The mostly studied are the models where only the nearest–neighbour (local) coupling is taken into account (see e.g. [8,9,10,12,13] and references therein). In other models the global coupling, i.e. when all elements of the system are coupled with each other, is assumed [15,16,17]. In lattices with the local coupling, as a rule, this is considered as a diffusional one [1,2,4,8,9,10,11,12,13]. Mainly, these models are used to analyse the spatio–temporal chaos, pattern formation, self–organization phenomena, etc. [18,19,20]

If all the elements are the same, then such lattices are called homogeneous. If, however, elements are different, then the lattice is not a homogeneous one, and its investigation is much more complex. With applied point of view the nonhomegeneity may be two–fold: either elements of the system have different maps, or they have functionally identical maps but with different control parameters. The nonhomogeneities are usually said to be *defects*. The relative position of defects may vary from a single localized inclusion to a space–periodic ordering.

In most of studies of coupled map lattices, the case of homogeneous lattices (and mainly numerically) has been addressed. However, it is clear, that the assumption of the homogeneity is a crude simplification. Therefore it is rather important to investigate the sensitivity of the lattice dynamics to certain non-homogeneities (see, e.g., [21,31] and references therein).

The present study is devoted to the spatially inhomogeneous 1D lattices (in fact, chains) of diffusively coupled piecewise–linear maps, which refer to some models of the one–dimensional excitable media [22]. The defects are determined as maps with different parameter values. Using the exact calculation of the Lyapunov exponents we analyse the behaviour of the lattice with a lo-calized defect and with the space–periodic nonhomegeneity. We also describe the structure of the phase space of such systems.

2 Models of spatially inhomogeneous excitable media

Here we consider the dynamics of two models of inhomogeneous active media: one–dimensional (annular) chain of coupled maps with periodic nonhomogeneity and with one localized defect. To introduce the notations, we remind that the term "map" is implied transformation of some interval I on itself: $T_a : I \longrightarrow I$, where $I = [\alpha, \beta] \in \mathbf{R}^1$, $x \mapsto G(x, a)$, $x \in I$, $G(x, a)$ is some function and a is a parameter. In terms of iterations k this may be written as $x(k + 1) = G(x(k), a)$, $x \in I$. The nonhomogeneity means nonuniformity of the parameter a along the chain of maps. The functions G are chosen in such a way that they model the one–component active medium.

2.1 Homogeneous medium

Consider the system of N diffusively coupled maps

$$x_n(k + 1) = G(x_n(k), \alpha, \gamma) + \varepsilon\Big(x_{n+1}(k) - 2x_n(k) + x_{n-1}(k)\Big) \tag{1}$$

with periodic boundary conditions, $x_{n+N}(k) \equiv x_n(k)$, where $n = 1, 2, \ldots, N$ being the discrete space coordinate, $k = 0, 1, 2, \ldots$ is the discrete time, $\varepsilon > 0$ is the (constant) diffusion coefficient, and $\alpha > 0$, γ are the parameters of the map. This choice of the coupling means that the spatial and temporal actions in map (1) occur simultaneously. Let G be the piecewise linear function:

$$G(x, \alpha, \gamma) = x + \alpha F(x) - \gamma = \begin{cases} (1 - 2\alpha)x - \gamma, & x < 1/2; \\[2mm] (1 - 2\alpha)x - \gamma + 2\alpha, & x > 1/2. \end{cases} \tag{2}$$

This function $G(x, \alpha, \gamma)$ is chosen in this form in order to represent (in the discretized version) the basic model of the one–component excitable medium, which is described by the Kolmogorov–Petrovsky–Piskunov type equation:

$$\frac{\partial x(z, t)}{\partial t} = D\frac{\partial^2 x(z, t)}{\partial z^2} + \Phi(x) \ .$$

This equation is frequently used in biophysical problems, in the combustion theory, in chemical kinetics (e.g. in the theory of the Belousov–Zhabotinsky reaction), solid-state physics, etc. (see, for example, [23,24,25,26]). One can see that the piecewise linear function $G(x, \alpha, \gamma)$ has a constant slope $(1 - 2\alpha)$. Depending on α the map $x \mapsto G(x, \alpha, \gamma)$ may exhibit qualitatively different behaviour. For $0 < \alpha < 1$ (see Fig.1a) it gives rise to a regular dynamics with one or two stable point, depending on the value of the parameter γ (points A

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and B in Fig.1a), which are attractive for almost all points of the phase space. For $\alpha > 1$ the slope of $G(x, \alpha, \gamma)$ is larger then one (see Fig.1b). That means, that for the finite motion of phase points the dynamics would be chaotic.

The uniform model (1) based on the coupled maps, which are described by functions (2), is known (see [22]). Following [22] consider this system as an N–dimensional piecewise–linear map $f : \mathbf{R}^N \rightarrow \mathbf{R}^N$. Because $G(x, \alpha, \gamma)$ is the piecewise–linear function with the constant derivative, the differential D_f of the map f is a matrix with constant coefficients. This easily allows us to find the Lyapunov exponents λ_i of such map. If ρ_s are the eigenvalues of D_f, then $\lambda_s = \ln |\rho_s|$. If among all the values ρ_s there exists an eigenvalue located outside the unit circle on the complex plane, then any trajectory of the map f is unstable. Otherwise the model has the regular dynamics. Thus, to know qualitatively the behavior of the system it is sufficient to find the location of the eigenvalues ρ_s on the complex plane.

To find ρ_s we should analyse the characteristic polynomial for the differential of the N–dimensional map, which may be expressed via the determinants of three–diagonal matrices of different dimension. For these determinants recurrent relations may be obtained. Taking the recurrent relations as difference equations with the given initial conditions we can find the solutions which are proportional to the Chebyshev polynomials with a linear function of the eigenvalues of D_f [22]. Using then the properties of the Chebyshev polynomials it can be shown that the characteristic polynomial may be reduced to the quadratic trinomial of a certain function ρ_s. Calculation of ρ_s and analysis of their location on the complex plane shows that the parametric space of the system (1), (2) has two regions with qualitatively different behavior (Fig.2).

(1) The region D_1, with the values of the parameters α and ε, which correspond to the regular dynamics of the model. This implies that the modules of all the eigenvalues of the differential of the map f are less than unity.

(2) The region D_2 where at least one of the eigenvalues ρ_s satisfies the condition $|\rho_s| > 1$. Generally, the system may have an infinite dynamics. However, if the dynamics is finite the system (1), (2) behaves chaotically [9,10,11].

The system (1), (2) is an uniform since all its elements are identical (one has the same map $x \mapsto G(x, \alpha, \gamma)$ for all the elements). Futher we will assume that each element of the system has the same map $x_i \mapsto G(x_i, \alpha_i, \gamma)$ but along the chain the values of the control parameter α_i may differ [1]. We assume that only two types of elements exist in the system. The corresponding values we denote as α and β. As for the case of uniform system we will use the periodic

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[1] In general, we can consider the case of different γ. But we will see that the dynamics of both uniform and nonuniform chains does not depends on γ.

boundary conditions, $x_{n+N}(k) \equiv x_n(k)$.

2.2 The model with spatially periodic nonhomegeneity

The periodic spatial nonhomegeneity implies in our case that the chain has elements with alternating values of the parameters in the map G, e.g. $\alpha, \beta, \alpha, \beta, \ldots$. This particular nonuniform model allows a complete analytical analysis.

Consider the system of coupled elements:

$$\begin{cases} x_n(k+1) = G(x_n(k), \alpha, \gamma) + \varepsilon\big(x_{n+1}(k) - 2x_n(k) + x_{n-1}(k)\big), & n \text{ is odd,} \\[2mm] x_n(k+1) = G(x_n(k), \beta, \gamma) + \varepsilon\big(x_{n+1}(k) - 2x_n(k) + x_{n-1}(k)\big), & n \text{ is even,} \end{cases} \tag{3}$$

where $x_{n+N}(k) \equiv x_n(k)$, N is even (to satisfy the periodic boundary condition), and the function G is given by (2). To analyse the dynamics of this system we calculate the Lyapunov exponents using the approach described above.

The differential of the N–dimensional map corresponding to the system (3) reads

$$\mathrm{D}_f \equiv Q_N = \begin{pmatrix} 1 - 2\varepsilon - 2\alpha & \varepsilon & 0 & \ldots & \varepsilon \\ \varepsilon & 1 - 2\varepsilon - 2\beta & \varepsilon & \ldots & 0 \\ 0 & \varepsilon & 1 - 2\varepsilon - 2\alpha & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & 0 & 0 & \ldots & 1 - 2\varepsilon - 2\beta \end{pmatrix}.$$

To find the eigenvalues ρ_s, $s = 1, 2, \ldots, N$ of the matrix Q_N, we calculate the determinant

$$\det(Q_N - \rho I_N) = \begin{vmatrix} 2z_1\varepsilon & \varepsilon & 0 & 0 & \ldots & \varepsilon \\ \varepsilon & 2z_2\varepsilon & \varepsilon & 0 & \ldots & 0 \\ 0 & \varepsilon & 2z_1\varepsilon & \varepsilon & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \varepsilon \\ \varepsilon & 0 & 0 & 0 & \ldots & 2z_2\varepsilon \end{vmatrix},$$

with z_1 and z_2 defined by the relations $1 - 2\varepsilon - 2\alpha - \rho = 2z_1\varepsilon$, $1 - 2\varepsilon - 2\beta - \rho = 2z_2\varepsilon$. Making expansion of the determinant we obtain:

$$\det(Q_N - \rho I_N) = 2z_1\varepsilon B_{N-1} - \varepsilon^2 B_{N-2} - 2\varepsilon^N - \varepsilon^2 \hat{B}_{N-2},$$

where

$$
B_N = \begin{vmatrix} 2z_1\varepsilon & \varepsilon & 0 & \ldots & 0 \\ \varepsilon & 2z_2\varepsilon & \varepsilon & \ldots & 0 \\ 0 & \varepsilon & 2z_1\varepsilon & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 2z_2\varepsilon \end{vmatrix}, \quad
\hat{B}_N = \begin{vmatrix} 2z_2\varepsilon & \varepsilon & 0 & \ldots & 0 \\ \varepsilon & 2z_1\varepsilon & \varepsilon & \ldots & 0 \\ 0 & \varepsilon & 2z_2\varepsilon & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 2z_1\varepsilon \end{vmatrix}.
$$

It is not hard to see that $\hat{B}_N(z_1, z_2)$ may be obtained from $B_N(z_1, z_2)$ exchanging the arguments $z_1 \leftrightarrow z_2$. This suggest the way of finding $B_N(z_1, z_2)$ using the recurrent relation. Expanding B_N (along the first line) we obtain:

$$
\begin{cases}
B_N = 2z_1\varepsilon B_{N-1} - \varepsilon^2 B_{N-2}, & N \text{ is even}; \\[2mm]
B_N = 2z_2\varepsilon B_{N-1} - \varepsilon^2 B_{N-2}, & N \text{ is odd}.
\end{cases}
$$

This allows us to express determinants of the odd order via determinants of the even one,

$$B_{2N+1} = \frac{B_{2N+2} + \varepsilon^2 B_{2N}}{2z_1\varepsilon}, \tag{4}$$

and obtain then the recurrent relation for the even determinants

$$B_{2N} = \varepsilon^2(4z_1z_2 - 2)B_{2N-2} - \varepsilon^4 B_{2N-4}. \tag{5}$$

One can treat Eq.(5) as a difference equation with the initial conditions

$$B_0 = 1, B_2 = \varepsilon^2(4z_1z_2 - 1). \tag{6}$$

Solving then (5), (6) together with (4), we arrive at:

$$
\begin{cases}
B_N(z_1, z_2) = \varepsilon^N U_N(z), & \text{for even } N; \\[3mm]
B_N(z_1, z_2) = \varepsilon^N \sqrt{\dfrac{z_2}{z_1}} U_N(z), & \text{for odd } N,
\end{cases}
$$

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where $U_N(z)$ is the second order Chebyshev polynomial. In turn, for $\hat{B}_N(z_1, z_2)$ we get:

$$\begin{cases} \hat{B}_N(z_1, z_2) = \varepsilon^N U_N(z), & \text{for even } N \text{ ;} \\[2ex] \hat{B}_N(z_1, z_2) = \varepsilon^N \sqrt{\dfrac{z_1}{z_2}}\, U_N(z), & \text{for odd } N. \end{cases}$$

Now, using the obtained relations for B_{N-1}, B_{N-2} and $\hat{B}_{N-2}$, we finally arrive at:

$$\det(Q_N - \rho I_N) = 2\varepsilon^N \Big(z U_{N-1}(z) - U_{N-2}(z) - 1\Big). \tag{7}$$

This allows us to find the eigenvalues of the differential of the map (3), (2). As follows from Eq.(7),

$$z U_{N-1}(z) - U_{N-2}(z) - 1 = 0, \tag{8}$$

or, $T_N(z) - 1 = 0$. Here we used the relation between the second order Chebyshev polynomials $U_N(z)$ and the first order $T_N(z)$. Since

$$T_N(z) = \frac{\left(z - \sqrt{z^2 - 1}\right)^N + \left(z - \sqrt{z^2 - 1}\right)^{-N}}{2},$$

where $z^2 = z_1 z_2$, then with $t = z - \sqrt{z^2 - 1}$ we obtain from Eq.(8):

$$\frac{1}{2}\left(t^N + \frac{1}{t^N}\right) - 1 = 0$$

or $\left(t^N - 1\right)^2 = 0$. Therefore, $t_s = e^{i\frac{2\pi}{N}s}$, $s = 1, 2, \ldots, N$, and $z_s = \cos(2\pi s/N)$, $s = 1, 2, \ldots, N$. Taking into account that $2z_1 \varepsilon \cdot 2z_2 \varepsilon = 4z^2 \varepsilon^2$, we obtain a simple equation:

$$(1 - 2\varepsilon - 2\alpha - \rho)(1 - 2\varepsilon - 2\beta - \rho) = 4\varepsilon^2 \cos^2\left(\frac{2\pi}{N}s\right).$$

Solving it with respect to ρ, we find the set of eigenvalues of the differential of the map (3), (2):

$$\rho_{1,2}^s = 1 - 2\varepsilon - \alpha - \beta \pm \sqrt{(\alpha - \beta)^2 + 4\varepsilon^2 \cos^2\left(\frac{2\pi}{N}s\right)}, \tag{9}$$

where $s = 1, 2, \ldots, N/2$. The dynamics of the system (3), (2) would be completely regular if all the values of ρ_s were within the unit circle on the complex

plane. Since all the $\rho^s_{1,2}$ are real, the condition of the regular dynamics is:

$$\begin{cases} 2\varepsilon + \alpha + \beta > \sqrt{4\varepsilon^2 + (\alpha - \beta)^2}, \\ \varepsilon + \dfrac{\alpha + \beta}{2} + \sqrt{\varepsilon^2 + \left(\dfrac{\alpha - \beta}{2}\right)^2} < 1. \end{cases}$$

The first inequality is always realized when $\varepsilon, \alpha, \beta > 0$. Thus, second one defines the restriction for the region with regular dynamics.

In its turn, in the region of the parametric space where the Lyapunov exponent $\lambda_s = \ln|\rho_s| > 0$ the condition $\varepsilon + (\alpha + \beta)/2 + \sqrt{\varepsilon^2 + (\alpha - \beta)^2/4} > 1$ is satisfied. Therefore, the boundary separating two regions with regular and chaotic dynamics is a surface in the three–dimensional parametric space $(\alpha, \beta, \varepsilon)$, which is given by the relation:

$$\varepsilon + \frac{\alpha + \beta}{2} + \sqrt{\varepsilon^2 + \frac{(\alpha - \beta)^2}{4}} = 1. \tag{10}$$

To illustrate the obtained results we show in Fig.3 cross–sections of this surface by the planes $\delta = \text{const}$, with $\delta = \beta - \alpha$ being the nonhomegeneity parameter. Denote D_1 the region with regular dynamics and D_2 the region where $\lambda_s > 0$. These two regions are separated by the boundary, defined (implicitly) by Eq.(10). In Fig.3 we also show by dashed line the corresponding boundary between D_1 and D_2 for the case of *homogeneous* ($\delta = 0$) system of coupled maps (Fig.2).

As it follows from the Fig.3, for $\delta > 1$ for all values of ε, α the model demonstrates the chaotic dynamics. For $\delta < 1$ the region D_1 arises where the system has regular dynamics. This region where $0 < \delta < 1$ is, in fact, a subregion of the corresponding region D_1 of the regular dynamics of the homogeneous model.

For $-1 < \delta < 0$ there exist such ε and α that the homogeneous model exhibits the chaotic dynamics, while the inhomogeneous still has the regular one. Note that the restriction $\alpha > -\delta$ corresponds to the condition $\beta > 0$. For $\delta < -1$ the region D_1 vanishes and for all ε, α the model (3), (2) demonstrates only the chaotic dynamics. As it also follows from Fig.3, two qualitatively different regimes make take place: α belongs to the region of the regular behavior, β to the region of chaos with $\lambda_s > 0$, with the global dynamics being regular. Similarly one can observe just the opposite case of the chaotic dynamics of the periodically inhomogeneous model. Realisation of the particular regime depends on the value of the diffusion constant ε.

Let us now turn to the case of spatially inhomogeneous model of diffusively coupled maps for the case of a single "defect". In this case among N element of the system only one element has the parameter β (see (2)), while the other $(N-1)$ elements have parameter α. Assume for simplicity that this element is located at $n = 1$. Then we can write for this model:

$$\begin{cases} x_n(k+1) = G(x_n(k), \alpha, \gamma) + \varepsilon\Big(x_{n-1}(k) - 2x_n(k) + x_{n+1}(k)\Big), \quad n = 2, 3, \ldots, N, \\ x_1(k+1) = G(x_1(k), \beta, \gamma) + \varepsilon\Big(x_N(k) - 2x_1(k) + x_2(k)\Big), \end{cases} \tag{11}$$

where $x_{n+N} \equiv x_n(k), n = 1, 2, \ldots, N$. Let the function $G(x, \alpha, \gamma)$ is still given by (2). Calculate now the differential of the corresponding map for the system (11), (2):

$$D_f \equiv Q_N = \begin{pmatrix} 1 - 2\varepsilon - 2\beta & \varepsilon & 0 & \ldots & \varepsilon \\ \varepsilon & 1 - 2\varepsilon - 2\alpha & \varepsilon & \ldots & 0 \\ 0 & \varepsilon & 1 - 2\varepsilon - 2\alpha \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & 0 & 0 & \ldots 1 - 2\varepsilon - 2\alpha \end{pmatrix} . \tag{12}$$

If we apply the same approach as previously, to find the Lyapunov exponents of the model (11), (2), we should have to get the roots of the polynomial of the $2N + 2$–th order. This can be done only numerically. Instead we will use somewhat different approach, which significantly facilitates the analysis.

Applying the estimate of the eigenvalues of the matrix (12) according to the Gershgorin's theorem [27], we may evaluate the location of the region of the regular dynamics. By this theorem, all the eigenvalues of the matrix $A = \{a_{ij}\}_{n\times n}$ which belongs to the union of the discs on the complex plane:

$$\rho_s \in \bigcup_{i=1}^{n} \Big\{z \in \mathbf{C} : |z - a_{ij}| \leq R_i'\Big\},$$

where $s = 1, 2, \ldots, N$ and R_i' is either a row almost–norm of the matrix A, $R_i' = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$, or a column almost–norm of the matrix A, $R_j' = \sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ij}|$. Since the differential of the map (12) is a real symmetric matrix then, (i) the row and column almost–norms are equal, and (ii) the eigenvalues of the matrix

D_f are real. Therefore, the Gershgorin's discs transform on the real axis as:

$$\rho_s \in \bigcup_{i=1}^{n} \left\{ z \in \mathbf{R} : |z - a_{ij}| \le R_i' \right\}, \quad s = 1, 2, \ldots, N .$$

Because $\varepsilon > 0$, from (12) we get:

$$R_i' = \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| = |\varepsilon| + |\varepsilon| = 2\varepsilon .$$

Moreover, since R_i' does not depend on i and in the differential of map D_f (12) there exist only two different diagonal elements, the eigenvalues of D_f belong to the union of the two intervals:

$$\left\{ z \in \mathbf{R} : |1 - 2\varepsilon - 2\alpha - z| \le 2\varepsilon \right\} \bigcup \left\{ z \in \mathbf{R} : |1 - 2\varepsilon - 2\beta - z| \le 2\varepsilon \right\}.$$

Then, taking into account that $\alpha, \beta, \varepsilon > 0$, we obtain the *upper* bound for the modules of the eigenvalues of D_f:

$$|\rho_s| \le \begin{cases} \beta + 2\varepsilon, & \alpha < \beta; \\ \\ \alpha + 2\varepsilon, & \alpha > \beta. \end{cases}$$

Thus, the region $\tilde{D}_1$ in the parametric space $(\alpha, \beta, \varepsilon)$, which satisfies the conditions

$$\tilde{D}_1 : \quad \begin{cases} \beta + 2\varepsilon < 1, & \text{if } \alpha < \beta; \\ \\ \alpha + 2\varepsilon < 1, & \text{if } \alpha > \beta; \end{cases} \tag{13}$$

corresponds to the regular dynamics of the inhomogeneous model (11), (2), while the region

$$\tilde{D}_2 : \quad \begin{cases} \beta + 2\varepsilon > 1, & \text{if } \alpha < \beta; \\ \\ \alpha + 2\varepsilon > 1, & \text{if } \alpha > \beta; \end{cases} \tag{14}$$

corresponds to the positive λ_s.

The regions $\tilde{D}_1$ and $\tilde{D}_2$ are shown in Fig.4. The region $\tilde{D}_1$ is the *lower* bound of the region with the regular dynamics, because it is obtained via the *upper* bound of the eigenvalues of the map differential. This means that any point of $\tilde{D}_1$ would correspond to the regular dynamics, while any point of $\tilde{D}_2$ would correspond either the regular dynamics, or (if the motion is finite) the chaotic one.

It is significant that the obtained estimate is applicable not only for the model with a single defect, but for a *wide class* of annular chains of maps with diffusive coupling (1), which have only two types of elements $\big($with different parameters α_i in the function $G(x_i, \alpha_i, \gamma)\big)$. This follows from the fact that the particular number of elements of each type and their mutual location is not important. Indeed, almost–norms of the differentials of the maps of all such systems are the same and the centres of the two possible intervals, which contain according to the Gershgorin's theorem, the eigenvalues of D_f, do coincide.

In the case when the two intervals, which we use to estimate ρ_s, do not intersect, there is, according to the Gershgorin's theorem, "clusterization" of the eigenvalues of the differential of the map. Namely, q eigenvalues ρ_s, $s = 1, 2, \ldots, q$, will belong to one interval, and $(N - q)$ eigenvalues ρ_s, $s = q+1, q+2, \ldots, N$ will belong to the other interval. Here q is the number of elements with parameter α, and, respectively, $(N - q)$ is the number of elements with parameter β. Unfortunately, this does not help to improve the estimate of the region with the regular dynamics.

Let us compare the estimate (13) with the exact result, that has been obtained above for the model with the periodic nonhomegeneity. Obviously, the latter model belongs to the class of models for which the this estimate holds true. To illustrate this we show the cross–sections $\delta = $ const of the region D_1, which corresponds to the regular dynamics of (3), (2), and of its estimate, the region $\tilde{D}_1$ (see Fig.5). From Fig.5 it follows that the estimate (13) of the region with the regular dynamics is rather adequate, however it does not show that for $-1 < \delta < 0$ the region of the regular dynamics of (3), (2) includes the region of the regular dynamics of the spatially homogeneous system.

Turn now to the map (11), (2), which refer to the case of a single defect and find the eigenvalues of the differential of map (12) numerically. One can write for the characteristic polynomial:

$$
\det(Q_N - \rho I_N) = \begin{vmatrix} 2z\varepsilon + 2(\alpha - \beta) & \varepsilon & 0 & \ldots & \varepsilon \\ \varepsilon & 2z\varepsilon & \varepsilon & \ldots & 0 \\ 0 & \varepsilon & 2z\varepsilon & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon & 0 & 0 & \ldots & 2z\varepsilon \end{vmatrix},
$$

where $2z\varepsilon = 1 - 2\varepsilon - 2\alpha - \rho$. Expanding $\det(Q_N - \rho I_N)$ in the first line we arrive at

$$
\det(Q_N - \rho I_N) = [2z\varepsilon + 2(\alpha - \beta)]B_{N-1} - 2\varepsilon^2 B_{N-2} - 2(-1)^N \varepsilon^N, \quad (15)
$$

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with

$$B_N = \begin{vmatrix} 2z\varepsilon & \varepsilon & 0 & \dots & 0 \\ \varepsilon & 2z\varepsilon & \varepsilon & \dots & 0 \\ 0 & \varepsilon & 2z\varepsilon & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2z\varepsilon \end{vmatrix}.$$

As previously, we use the first line factorization of B_N to obtain the recurrent relation, $B_N = 2z\varepsilon B_{N-1} - \varepsilon^2 B_{N-2}$. Using the initial conditions, $B_1 = 2z\varepsilon$, $B_2 = \varepsilon^2(4z^2 - 1)$, we obtain:

$$B_N = \varepsilon^N U_N(z), \tag{16}$$

where $U_N(z)$ is the second order Chebyshev polynomial. Taking into account Eq.(16) and representation of $U_N(z)$,

$$U_N(z) = \frac{(z + \sqrt{z^2 - 1})^{N+1} - (z - \sqrt{z^2 - 1})^{N+1}}{2\sqrt{z^2 - 1}} = \frac{\left(\frac{1}{t}\right)^{N+1} - t^{N+1}}{\frac{1}{t} - t},$$

where $t = z - \sqrt{z^2 - 1}$, we recast Eq.(15) into the following form:

$$\det(Q_N - \rho I_N) = 2\varepsilon^N \left[\frac{t^{2N+1} + 1}{2t^N} + \left(\frac{\alpha - \beta}{\varepsilon}\right) \frac{1 - t^{2N}}{(1 - t^2)t^{N-1}} - (-1)^N \right].$$

Thus, the characteristic equation for the differential of the map (12) reads

$$t^{2N+2} + 2\left(\frac{\alpha - \beta}{\varepsilon}\right)t^{2N+1} - t^{2N} - 2(-1)^N t^{N+2} + 2(-1)^N t^N + t^2 - 2\left(\frac{\alpha - \beta}{\varepsilon}\right)t - 1 = 0.$$

Finding numerically the roots of this equation, we obtain ρ_s. Comparing ρ_s with unity for different sets of α, β, ε, for the annular chain of diffusively coupled maps with a single defect we find the regions D_1 and D_2 with qualitatively different dynamics. In Fig.6 we show cross–sections of these regions by the planes with the fixed values of $\delta = \text{const}$, where $\delta = \beta - \alpha$. The estimate (13) of the region with the regular dynamics is marked by the dashed line. As in the case of the periodic nonhomegeneity, the actual region for positive δ is larger than that predicted by the estimate. However for the negative δ the estimate is in a perfect agreement with the numerical results as well as with the results corresponding to the homogeneous chain. Thus, if one of the elements of the chain had the parameter smaller than that of the others, there would be no impact on the global dynamics of the system. On the other hand, if the parameter which refers to the defect had larger value than that of the other

elements, this would diminish the region of the regular dynamics as compared to the case of the uniform system.

3   Conclusions

We investigated the dynamics of diffusively coupled one–dimensional maps, which model one–dimensional excitable medium with space nonhomegeneity. We considered different types of the nonhomogeneities. For all systems, with the spatial nonhomegeneity, which may be described in terms of two types of elements with different values of the control parameter, we found *analytically* a lower bound of the region in the parametric space where the ensemble exhibits the regular dynamics. This estimate has been obtained by localizing the eigenvalues of the differential of the map. For two different model of this class we also obtained exact analytical and numerical results which completely characterize the dynamics of the system.

We observed that all three approaches (the approximate analytical analysis of the chain with two types of elements (13), the exact solution for the spatially periodic defects (§2.2) and numerical study (§2.3) of the model with a single defect) show the possibility of two qualitatively different types of the ensemble behavior. In the first case the system demonstrates a regular global behavior when one value of its parameter belongs to the region with the regular dynamics and the other one belongs to the region with the unstable dynamics. Vice versa, in the second case the global dynamics occurs to be unstable, and if the motion is finite — chaotic. The particular type of the chain behavior depends on the diffusive parameter ε. Obviously, if ε is large enough, the global dynamics would not be regular.

We wish to stress that according to our knowledge, the present study is the first attempt of the *analytical* description of the dynamics of inhomogeneous systems of coupled maps. These systems, being of significant importance from the point of view of applications, may exhibit a rich physical behavior, giving rise to a lot of new and interesting problems. One can mention such problems as properties of the wave travelling along the chain and influence of the defects on these motions, dynamics of the ensemble with time–dependent parameters, controllability conditions, synchronization and many others (see, e.g., [29,30,31,18,19,20]). These problems, not discussed here will be addressed elsewhere [28,32].

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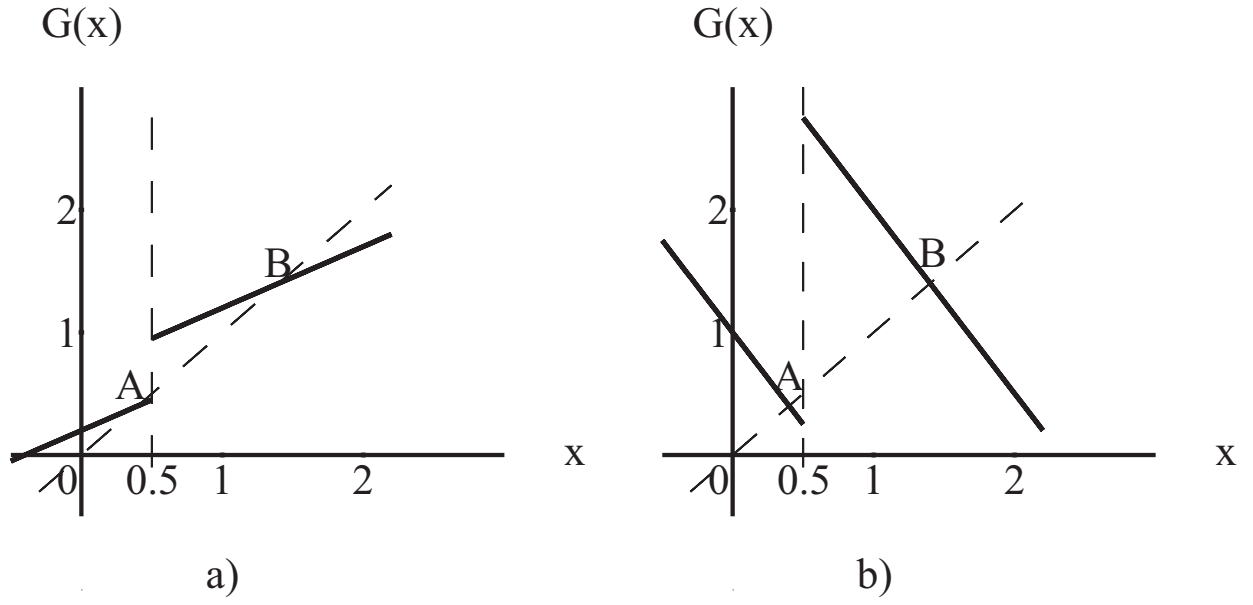[32] A.Loskutov, S.D.Rybalko, R.Seleznev.— To be published.

Fig. 1. The map $x \mapsto G(x, \alpha, \gamma)$ for $\alpha = 0.25$, $\gamma = -0.2$ (a) and $\alpha = 1.25$, $\gamma = -1$ (b).
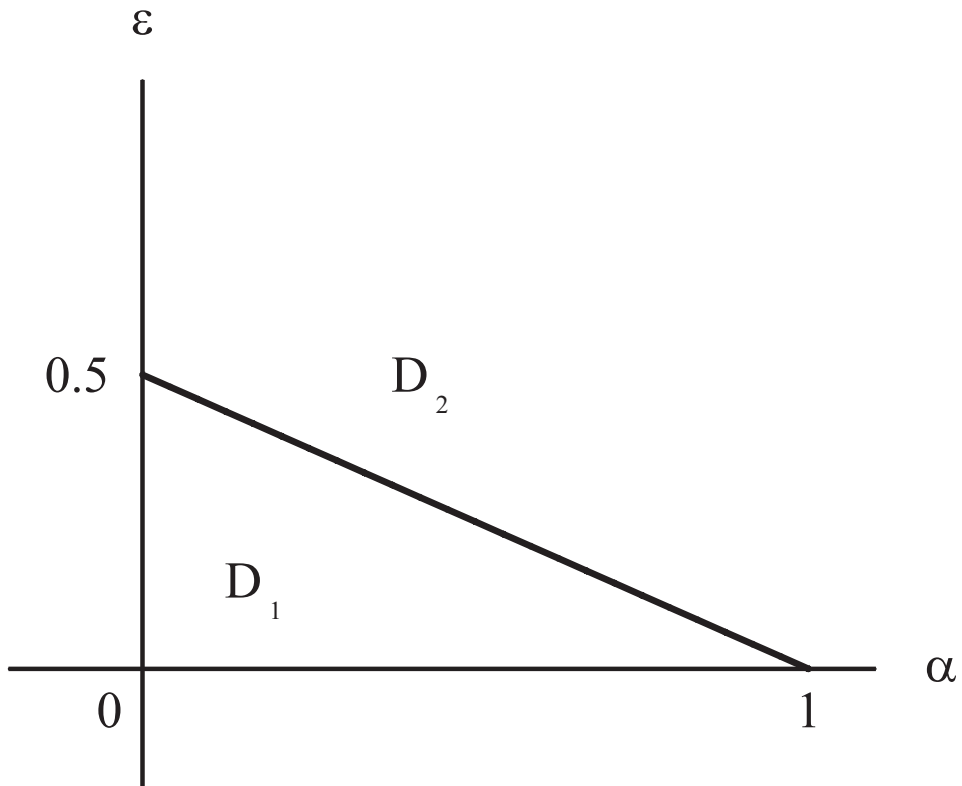
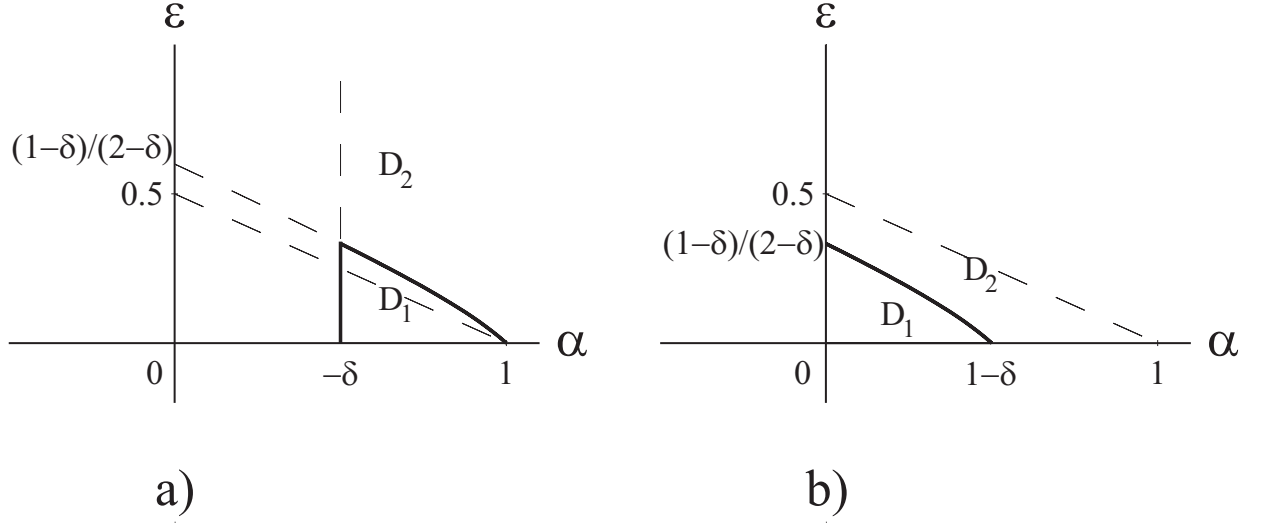Fig. 2. The parametric space of the homogeneous annular chain of the diffusively coupled maps (1), (2).

Fig. 3. Cross–sections by the planes $\delta = \mathrm{const}$ of the parametric space of the model (2), (3) for $-1 < \delta < 0$ (a) and $0 < \delta < 1$ (b).
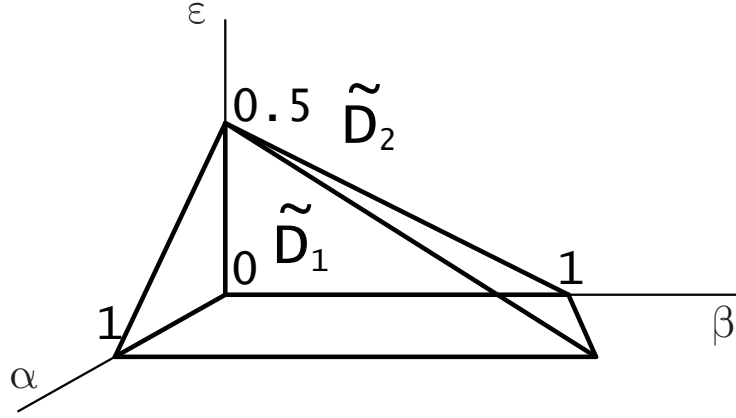
Fig. 4. The estimate of the region with the regular dynamics for the inhomogeneous chains with two types of elements in the parametric space $(\alpha, \beta, \varepsilon)$.
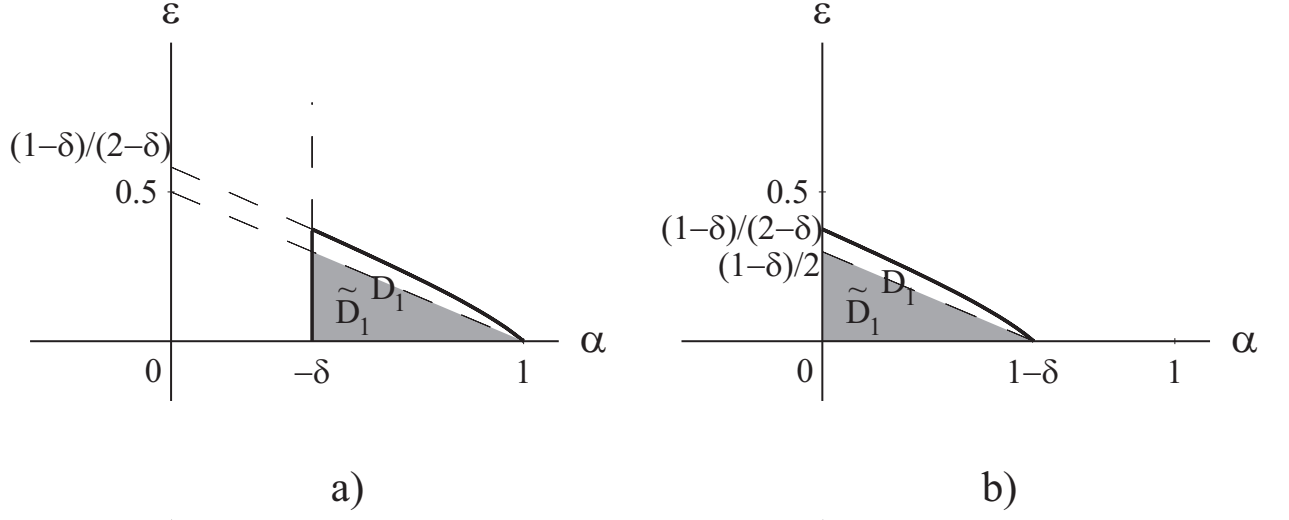
Fig. 5. The estimate and the exact result for the region with the regular dynamics for the model with periodic nonhomegeneity for $-1 < \delta < 0$ (a) and $0 < \delta < 1$ (b).
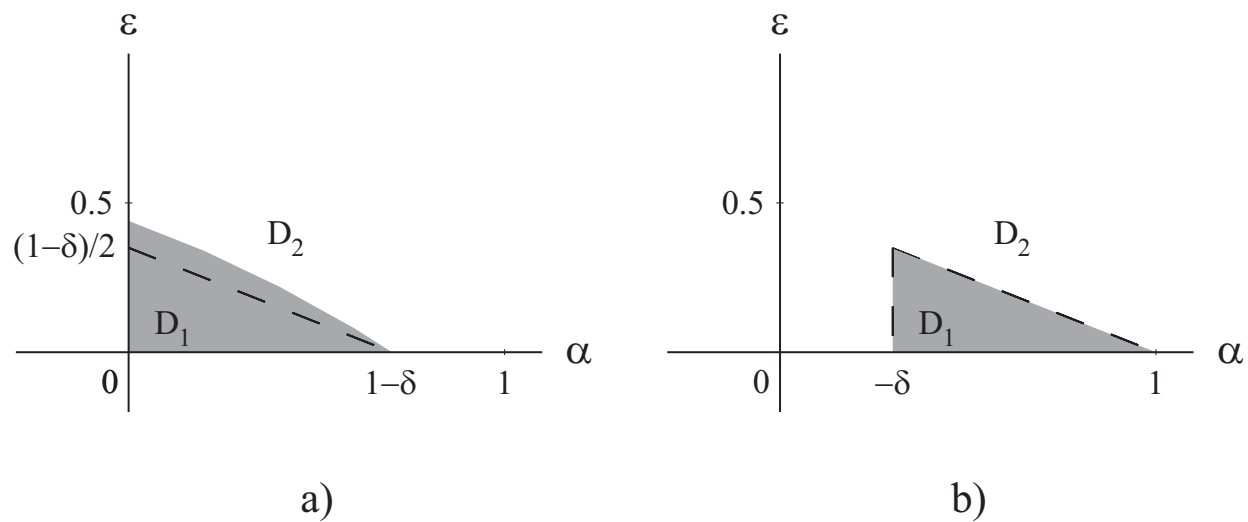
Fig. 6. Cross–sections by the planes $\delta = \mathrm{const}$ of the parametric space for the chain of $N = 20$ elements with a single defect for $\delta = -0.3$ (a) and $\delta = 0.3$ (b).